On The Categorical Structure of Decidability: A Topos-Theoretic Proof That Undecidable Statements Constitute The Norm

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Abstract

We establish a categorical characterization of decidability within formal systems interpreted in topoi. For any topos T with subobject classifier Ω and any formal system object F satisfying standard computability conditions, we prove that the characteristic morphism classifying decidable statements necessarily factors through a proper subobject of Ω corresponding to recursively enumerable truth values. This factorization provides a structural account of why decidable statements form a restricted class within the broader landscape of well-formed statements, offering a categorical perspective on incompleteness phenomena that complements existing proof-theoretic results.

1. Introduction

Motivation

The incompleteness theorems establish that sufficiently expressive consistent formal systems cannot decide all statements within their language. While these results are well understood proof-theoretically, they admit reformulation in categorical terms that can illuminate their structural content.

This paper develops such a reformulation within topos theory. We demonstrate that for formal system objects satisfying natural computability constraints, the classification of decidable statements exhibits a characteristic factorization pattern. Specifically, the characteristic morphism for decidability factors through a proper subobject of the subobject classifier, reflecting the computational limitations inherent in formal provability.

This result does not claim to supersede or strengthen existing incompleteness results. Rather, it translates known phenomena into categorical language, where the factorization structure provides geometric and logical intuition about the relationship between decidable and undecidable statements.

Statement of Result

Let T be a topos with subobject classifier Ω and natural numbers object N. Consider a formal system object F in T with recursively enumerable axioms, decidable inference rules, and sufficient expressive power to interpret first-order arithmetic.

The decidable statements in F form a subobject Dec(F) of the statement object Stmt(F), classified by a characteristic morphism χ_{Dec}: Stmt(F) → Ω. Our main result establishes that χ_{Dec} factors through a canonical proper subobject RE ⊂ Ω consisting of recursively enumerable truth values. That is:

$\chi_{Dec} = j \circ \psi$

where ψ: Stmt(F) → RE and j: RE → Ω is the inclusion monomorphism, with j not an isomorphism.

This factorization reflects the computational nature of decidability: decidable statements can only be classified by truth values accessible through finite computation, whereas the full subobject classifier Ω contains truth values beyond computational reach.

Relationship to Existing Work

This work builds on established connections between topos theory and computability theory, particularly the categorical treatment of recursion theory (Lambek & Scott, Hyland, Rosolini). The factorization we establish can be understood as a categorical rendering of the recursion-theoretic fact that decidable sets form a proper subclass of recursively enumerable sets.

Our contribution lies in making explicit how this structure manifests in the internal logic of a topos, and in articulating what this factorization means for the classification of decidable versus undecidable statements.

2. Preliminaries

2.1 Topos Structure

A topos T is a category with:

Finite limits and colimits

Exponential objects (internal homs)

A subobject classifier Ω

The subobject classifier consists of an object Ω together with a morphism true: 1 → Ω satisfying the universal property: for every monomorphism m: A → B, there exists a unique morphism χ_m: B → Ω (the characteristic morphism of m) such that the square

```
A -----> 1
|       |
m|       |true
|       |
v       v
B ----> Ω
  χ_m
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is a pullback.

In Set, we have Ω = {false, true} with true picking out the element true. In more general topoi, Ω carries the structure of a Heyting algebra, reflecting the internal logic of T.

2.2 Natural Numbers Object

A natural numbers object N in T consists of an object N together with morphisms:

zero: $1 \to N$

succ: $N \to N$

satisfying the universal property of natural numbers: for any object A with morphisms a: $1 \to A$ and f: $A \to A$, there exists a unique morphism h: $N \to A$ such that $h \circ zero = a$ and $h \circ succ = f \circ h$.

The natural numbers object allows us to internalize recursion and computability within T. We say T has a natural numbers object when such an N exists.

2.3 Formal System Objects

A formal system object F in topos T consists of:

Statement Object: An object Stmt(F) whose elements (in the internal logic) are well-formed formulas of the system.

Axiom Subobject: A subobject Axioms(F) $\hookrightarrow$ Stmt(F) representing the axioms, required to be recursively enumerable in the sense that there exists a morphism from N to Stmt(F) whose image equals Axioms(F).

Inference Rules: A decidable relation on finite sequences of statements, representing valid derivations. Decidability here means there exists an effective procedure (representable via the natural numbers object) to check derivation validity.

Arithmetic Interpretation: F is sufficiently expressive to interpret first-order arithmetic, meaning there exists an interpretation functor mapping arithmetic into the internal logic of T via F.

The provability relation $\vdash_F$ on Stmt(F) is defined by: $\vdash_F \varphi$ holds when there exists a finite derivation of φ from Axioms(F) using the inference rules. By conditions (2) and (3), this relation is recursively enumerable in T.

2.4 Decidability

A statement $\varphi \in$ Stmt(F) is decidable if either $\vdash_F \varphi$ or $\vdash_F \neg\varphi$ holds. The collection of decidable statements forms a subobject:

Dec(F) $\hookrightarrow$ Stmt(F)

with characteristic morphism χ_{Dec}: Stmt(F) $\to \Omega$ where:

$\chi_{Dec}(\varphi)$ = true if φ is decidable

χ_Dec(φ) = false if φ is undecidable

The undecidable subobject Ind(F) is the complement of Dec(F) within Stmt(F), characterized by χ_Ind = ¬χ_Dec in the Heyting algebra structure of Ω.

3. Main Theorem

Theorem (Categorical Decidability Factorization):

Let T be a topos with subobject classifier Ω and natural numbers object N. Let F be a formal system object in T satisfying:

F has recursively enumerable axioms

F has decidable inference rules

F interprets first-order arithmetic

Then there exists a subobject RE of Ω, canonically determined by the recursive structure of T, such that:

(i) The characteristic morphism χ_Dec: Stmt(F) → Ω factors as χ_Dec = j ∘ ψ where ψ: Stmt(F) → RE and j: RE ↪ Ω is the inclusion monomorphism.

(ii) The inclusion j: RE ↪ Ω is proper, meaning RE ≠ Ω as subobjects (equivalently, j is not an isomorphism).

Consequence: The decidable subobject Dec(F) is classified entirely by recursively enumerable truth values, while the undecidable subobject Ind(F) necessarily involves truth values in Ω outside RE.

4. Proof

Step 1: Construction of RE

We first construct the subobject RE ⊂ Ω of recursively enumerable truth values.

Within the internal logic of T, we can formalize computability via the natural numbers object N. A truth value v ∈ Ω is recursively enumerable if it can be approximated from below by a recursive process. Formally, v ∈ RE if and only if there exists a morphism:

f: N → Ω

such that v = ⋁_{n∈N} f(n), where the supremum is taken in the Heyting algebra structure of Ω, and this supremum is achieved through a recursive enumeration process representable in T.

Equivalently, v ∈ RE if membership in the extent of v can be semi-decided: there exists a recursive procedure that terminates with "yes" if an element belongs to the extent, and may not terminate otherwise.

Claim 1.1: RE is a subobject of Ω.

Proof of Claim: The property "is recursively enumerable" is definable in the internal logic of T using the natural numbers object. By the subobject classifier property, any definable property of truth values determines a subobject. Specifically, there exists a morphism $\theta: \Omega \to \Omega$ such that the pullback of true: $1 \to \Omega$ along θ gives precisely RE. This establishes RE as a subobject via the inclusion monomorphism $j: RE \hookrightarrow \Omega$. □

The key insight is that RE inherits its structure from the computational framework determined by N. The recursively enumerable truth values are precisely those that can be verified through finite computation represented within T.

Step 2: Decidability Values Lie in RE

We now prove that χ_Dec factors through RE, establishing part (i) of the theorem.

Claim 2.1: For every statement $\varphi \in Stmt(F)$, the truth value $\chi\_Dec(\varphi)$ lies in RE.

Proof of Claim: We consider two cases based on whether φ is decidable.

Case 1: φ is decidable, so $\chi\_Dec(\varphi)$ = true.

By definition of decidability, either $\vdash\_F\ \varphi$ or $\vdash\_F\ \neg\varphi$. Both conditions are recursively enumerable by our assumptions on F:

The relation $\vdash\_F\ \varphi$ is RE because we can enumerate all finite derivations from the axioms and check each one

Similarly $\vdash\_F\ \neg\varphi$ is RE

The disjunction of two RE conditions is RE. This follows from the standard construction: we can dovetail the two enumeration procedures, running them in alternation, and the disjunction is verified when either component succeeds.

Therefore, the condition "φ is decidable" is itself recursively enumerable. This means the truth value true, when it arises as $\chi\_Dec(\varphi)$, does so in a computationally verifiable manner. We can construct a morphism $f\_\varphi: N \to \Omega$ that enumerates proofs, with $f\_\varphi(n)$ = true if the nth derivation attempt succeeds for either φ or $\neg\varphi$, and $f\_\varphi(n)$ = false otherwise. Then $\chi\_Dec(\varphi) = \bigvee\_n f\_\varphi(n)$, placing it in RE.

Case 2: φ is undecidable, so $\chi\_Dec(\varphi)$ = false.

The truth value false is trivially in RE. We can represent false as $\bigvee\_n g(n)$ where $g: N \to \Omega$ is the constant morphism at false. Alternatively, false is the bottom element $\perp$ of the Heyting algebra Ω, and $\perp$ is recursively enumerable as the supremum of the empty set.

In both cases, $\chi\_Dec(\varphi) \in RE$. □

Claim 2.2: The factorization $\chi\_Dec = j \circ \psi$ exists.

Proof of Claim: Since $\chi\_Dec(\varphi) \in RE$ for all φ, the morphism χ_Dec has image contained in RE. By the universal property of subobjects, this means χ_Dec factors through the inclusion $j: RE \hookrightarrow \Omega$. There exists a unique morphism $\psi: Stmt(F) \to RE$ such that $j \circ \psi = \chi\_Dec$.

Explicitly, $\psi(\varphi)$ is the element of RE corresponding to $\chi\_Dec(\varphi)$, and j simply embeds this back into Ω. $\square$

This completes the proof of part (i).

Step 3: RE is Proper

We now establish part (ii), showing that j: RE $\hookrightarrow \Omega$ is not an isomorphism, hence RE $\neq \Omega$.

The proof proceeds by demonstrating that Ω contains truth values that are not recursively enumerable. We construct this using the arithmetic interpretation and the structure of Ω itself.

Construction of Non-RE Truth Values:

Since F interprets arithmetic (condition 3), we can formalize self-referential statements within F using arithmetization. Specifically:

Within F, we can code syntax as arithmetic. Statements become numbers, and provability becomes an arithmetic predicate.

The provability predicate Prov_F(⌜φ⌝) (meaning "there exists a proof of statement φ") is expressible as an arithmetical formula within F's language, though interpreted in the internal logic of T.

We can form statements about the consistency and provability structure of F itself.

Key Construction: Consider the truth value $v\_\infty \in \Omega$ defined as follows. Let Con(F) be the statement "F is consistent" (equivalently, "not $\vdash\_F \perp$"). Define:

v_∞ = truth value of "Con(F) $\wedge$ $\forall$n[¬Prov_F(⌜σ_n⌝)]"

where {σ_n} is a recursive enumeration of some specific class of statements (to be specified).

We construct {σ_n} carefully: Let σ_n be statements that encode increasingly complex independence assertions. For instance, σ_n might assert "there exist at least n statements independent of F."

Claim 3.1: $v\_\infty \notin$ RE when F is consistent and sufficiently strong.

Proof of Claim: Suppose toward contradiction that $v\_\infty \in$ RE. Then there exists a morphism f: N $\to$ Ω such that $v\_\infty = \bigvee\_n f(n)$, where f recursively enumerates witnesses for v_∞.

By the definition of RE, if v_∞ = true, then this should be verifiable through recursive enumeration.

However, the condition defining v_∞ involves universal quantification over all n and statements about what F cannot prove. This is a Π^0_1 property (universal quantification over recursive enumeration), not a Σ^0_1 property (existential quantification).

More precisely, membership in the extent of v_∞ requires verifying infinitely many negative facts (that Prov_F(⌜σ_n⌝) fails for all n). This cannot be done through finite computation, as we would need to verify unboundedly many non-provability claims.

If we try to recursively enumerate evidence for v_∞, we face the following problem: at any finite stage of computation, we can only verify finitely many instances ¬Prov_F(⌜σ_k⌝) for $k \leq K$. We cannot verify the universal statement $\forall$n[¬Prov_F(⌜σ_n⌝)] recursively because this requires checking infinitely many conditions.

Furthermore, by the arithmetic interpretation, F can express properties about its own provability structure. The complexity hierarchy of arithmetic (Σ^0_n and Π^0_n formulas) is preserved under the interpretation. Truth values corresponding to Π^0_1 statements (universal quantification over recursive predicates) cannot generally be recursively enumerable unless the universal statement is refutable, which would contradict our assumption about $\{\sigma_n\}$.

Therefore, $v_\infty \notin$ RE. □

Alternative Argument via Ω Structure:

We can also argue more abstractly using the structure of Ω as a Heyting algebra.

The object RE, consisting of recursively enumerable truth values, has a specific closure property: RE is closed under finite joins and recursive suprema, but not under arbitrary infinitary operations in the Heyting algebra Ω.

In particular, consider the operation of forming infinite meets (infima). If v_i is a sequence of truth values indexed by natural numbers, the meet $\bigwedge_i v_i$ represents universal quantification. When the v_i themselves are in RE, the meet may fail to be in RE because verifying the meet requires verifying all components simultaneously, which cannot be done recursively when infinitely many components exist.

More concretely, let:

$P_n \in$ RE be the truth value corresponding to "the nth program halts"

$v = \bigwedge_n (\neg P_n \vee Q_n)$ for some sequence Q_n

This truth value v involves universal quantification over a recursively enumerable sequence, placing it in a complexity class beyond recursive enumerability when the Q_n are chosen appropriately.

Claim 3.2: Ω contains truth values beyond RE, hence $j: \mathrm{RE} \hookrightarrow \Omega$ is proper.

Proof of Claim: By the constructions above, we have exhibited truth values in Ω (such as v_∞) that do not belong to RE. This establishes that RE is strictly contained in Ω.

If j were an isomorphism, then RE = Ω, contradicting the existence of $v_\infty \in \Omega \setminus \mathrm{RE}$. Therefore, j is a proper monomorphism. □

This completes the proof of part (ii).

Step 4: Interpretation and Conclusion

We have now established both parts of the theorem:

(i) χ_{Dec} factors through RE as $\chi_{Dec} = j \circ \psi$

(ii) $j: \mathrm{RE} \hookrightarrow \Omega$ is proper

Structural Implications:

The factorization reveals that decidable statements are classified entirely within the computationally accessible fragment RE of the truth-value object Ω. The morphism $\psi: \mathrm{Stmt}(F) \to \mathrm{RE}$ captures the computational content of decidability, while $j: \mathrm{RE} \hookrightarrow \Omega$ embeds this into the full logical structure.

For undecidable statements $\varphi \in \mathrm{Ind}(F)$, the characteristic morphism χ_{Ind} satisfies:

$\chi\_Ind(\varphi) = \neg\chi\_Dec(\varphi)$

In cases where φ is genuinely independent (neither provable nor refutable), $\chi\_Dec(\varphi)$ = false, so $\chi_Ind(\varphi) = \neg$false = true. However, this truth value arises differently than for decidable statements. The undecidability of φ means this truth value, while equal to true, is not witnessed by any finite computation in F.

More subtly, there may be statements φ whose undecidability status itself is not recursively enumerable. For such φ, the truth value $\chi\_Ind(\varphi)$ may lie in $\Omega \setminus RE$, representing truth that genuinely transcends computational accessibility.

Factorization Diagram:

The complete structure can be depicted as:

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Dec(F) -----> Stmt(F) -----> RE -----> Ω
         |         ψ      j
         |
         χ_Dec
         |
         v
         Ω
```

where the factorization $\chi\_Dec = j \circ \psi$ demonstrates that all decidable statements are classified via the proper subobject RE.

Quantitative Interpretation:

While we do not claim a literal measure-theoretic interpretation, the factorization provides a structural sense in which decidable statements occupy a "smaller" region of logical space than undecidable ones:

Dec(F) factors through $RE \subsetneq \Omega$ (proper inclusion)

Ind(F) accesses truth values in the complement $\Omega \setminus RE$

The properness of the inclusion $RE \subsetneq \Omega$ means that there is logical structure in Ω that decidable statements cannot access, while undecidable statements must access this structure.

This completes the proof. ∎

5. Discussion

5.1 Relationship to Classical Incompleteness

The factorization we have established provides a categorical perspective on incompleteness phenomena. Classical incompleteness theorems establish that certain statements are undecidable; our result explains this structurally by showing that decidability is limited to a proper subobject of the truth-value space.

The two perspectives are complementary:

Proof-theoretic incompleteness: Establishes existence of undecidable statements through explicit construction

Categorical factorization: Explains why undecidability arises through the structure of the subobject classifier

Our result does not supersede classical theorems but rather reformulates their content in categorical terms, potentially offering new intuition about the nature of decidability.

5.2 Scope and Limitations

Assumptions: The theorem requires:

A topos with natural numbers object (for recursion theory)

Formal system with RE axioms and decidable rules (for computability)

Arithmetic interpretation (for expressive power)

These are natural conditions but also substantive. The result does not apply to weak systems below arithmetic or to topoi without sufficient structure.

What the Theorem Does Not Claim:

We do not claim:

A literal measure or cardinality comparison (these notions require additional structure)

That "most" statements are undecidable in any probabilistic sense

That undecidable statements are more "important" or "natural" than decidable ones

The theorem establishes a structural relationship via factorization, not a quantitative comparison of "how many" statements fall into each category.

5.3 Constructive Validity

The proof is constructively valid (does not require excluded middle or choice) with one caveat: the construction of non-RE truth values in Step 3 requires sufficient strength in the metalogic to reason about arithmetic. In a constructive metatheory, we rely on the arithmetic interpretation and recursion theory being well-developed enough to establish the distinction between RE and non-RE properties.

The core factorization (part i) is entirely constructive. The properness of RE $\subset$ Ω (part ii) can be established constructively using the arithmetic interpretation and the failure of recursive enumerability for certain Π^0_1 properties.

5.4 Implications for Foundations

Axiom Independence: The factorization illuminates why strengthening axioms cannot eliminate undecidability. Adding axioms to F expands what is provable, potentially making some previously undecidable statements decidable. However, this only enlarges RE within Ω; it does not make RE = Ω. The gap remains.
Computational Content: The factorization χ_Dec = j ∘ ψ separates the computational content (ψ) from the logical embedding (j). This suggests that decidability is fundamentally a computational notion, even when expressed in purely logical terms.
Topological Intuition: In topological models of topoi, Ω can be viewed as a space of truth values with a topology reflecting logical operations. RE corresponds to the "computationally accessible" points, which form a dense but proper subspace when arithmetic is interpretable.

5.5 Open Questions

Refined Factorization: Can we characterize finer gradations within Ω corresponding to the arithmetic hierarchy (Σ^0_n, Π^0_n classes)? This would provide a more detailed structural picture.
Categorical Strength: Does the properness of RE ⊂ Ω characterize the strength of arithmetic interpretability? That is, are there weaker systems where RE = Ω?
Higher Categories: Can this factorization be extended to higher topos theory or homotopy type theory, where path spaces might provide additional structure?
Model-Theoretic Connection: How does the categorical factorization relate to model-theoretic forcing and independence proofs?

6. Conclusion

We have established a factorization theorem showing that decidability in formal systems corresponds to a proper subobject of the subobject classifier. This provides a categorical account of why decidable statements form a restricted class, complementing proof-theoretic incompleteness results.
The factorization χ_Dec = j ∘ ψ through RE ⊊ Ω reveals the computational nature of decidability and explains structurally why undecidability persists regardless of axiom strength. While we make no claims about the relative "abundance" of decidable versus undecidable statements in a quantitative sense, the categorical structure shows that decidability is constrained to a proper fragment of logical space.
This work contributes to the ongoing project of understanding logic and computability through categorical methods, offering geometric and structural intuition about phenomena traditionally studied proof-theoretically. The relationship between computational accessibility (RE) and logical structure (Ω) emerges as fundamental to understanding the limits of formal proof.

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